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CONCERNING A CLASS OF SOLUTIONS OF THE DIRAC EQUATION

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ABSTRACT: Solutions of the Dirac equation are developed for the case of a sinusoidal field and the case in which the external field consists of polarized waves travelling in a certain direction and having a countable spectrum from the standpoint of frequency and initial phases.

1. The case of a sinusoidal field. - 2. Solution for the case in which the external field consists of polarized waves travelling in a certain direction having a *countable* spectrum from the standpoint of frequency and intial phases. /250[1]

1. The Case of a Sinusoidal Field

Let the scalar potential of the external field acting on the relativistic quantum electron equal zero and let the vector potential be

$$A = a\cos 2\pi\nu\left[t - \frac{nx}{c} + \alpha\right] = a\cos\varphi \text{ with } \varphi = 2\pi\nu\left[t - \frac{nx}{c} + \alpha\right];$$

ν here is a constant (frequency), t time, c the velocity of light, n a unit vector (which indicates the direction of propagation of an electromagnetic wave associated with A); x denotes the vector proceeding from the initial point of the rectangular Cartesian coordinate system, selected so as to be stationary, according to the variable point, a system for which the above vector potential is prescribed. $nx = (n, x) = xn$ is the sign of the scalar product;

[1] Numbers in the margin indicate pagination in the foreign text.

a the constant vector with real components perpendicular to n (so that $(a, n) = 0$); and lastly, α^0 is a constant.

Let the components of the vectors x, A, a, n ...be respectively

$$x_1, x_2, x_3; \quad A_1, A_2, A_3; \quad a_1, a_2, a_3; \quad n_1, n_2, n_3; \ \ldots$$

In our case the external field represents a sinusoidal electromagnetic wave of fixed direction and fixed frequency. The Dirac equation for the function ψ (x, t), which here describes the state of the electron in the variables x, t ~~as~~ is follows:

$$\left\{\frac{i\hbar}{c}\frac{\partial}{\partial t} + \left(\alpha, -i\hbar\frac{\partial}{\partial x} + \frac{e}{c}A\right) + \varrho_3 m c\right\}\psi(x, t) = 0. \tag{1}$$

In this equation $\hbar$ is the Planck constant divided by 2π, $\dfrac{\partial}{\partial x}$ a vector with the components $\dfrac{\partial}{\partial x_1}, \dfrac{\partial}{\partial x_2}, \dfrac{\partial}{\partial x_3}$; α is a matrix vector $= \varrho_1\sigma$; σ has the components $\sigma_1, \sigma_2, \sigma_3$, $\sigma_1, \sigma_2, \sigma_3$, $\varrho_1, \varrho_2, \varrho_3, \alpha_1, \alpha_2, \alpha_3$ are the known Dirac matrices; e is the charge of the electron; m is the mass.

If the operator on the right in the braces of equation (1) is multiplied by

$$\frac{i\hbar}{c}\frac{\partial}{\partial t} - \left(\alpha, -i\hbar\frac{\partial}{\partial x} + \frac{e}{c}A\right) - \varrho_3 m c, \tag{2}$$

a new operator arises

$$\Big\{\left(\frac{i\hbar}{c}\frac{\partial}{\partial t}\right)^2 - \left(\sigma, -i\hbar\frac{\partial}{\partial x} + \frac{e}{c}A\right)^2 - m^2c^2$$
$$-\varrho_1\left[\left(\frac{i\hbar}{c}\frac{\partial}{\partial t}\right)^2\left(\sigma, -i\hbar\frac{\partial}{\partial x} + \frac{e}{c}A\right) - \left(\sigma, i\hbar\frac{\partial}{\partial x} + \frac{e}{c}A\right)\left(\frac{i\hbar}{c}\frac{\partial}{\partial t}\right)\right]\Big\}.$$

It can be shown by means of transformations similar to those given by Dirac (for the reverse sequence of factors) (page 288)[1], that the last operator equals

$$\left(\frac{i\hbar}{c}\frac{\partial}{\partial t}\right)^2 - \left(-i\hbar\frac{\partial}{\partial x} + \frac{e}{c}A\right)^2 - m^2c^2 - \frac{\hbar e}{c}(\sigma, H) + i\varrho_1\frac{\hbar e}{c}(\sigma, E)$$

H denotes the magnetic and E the external electric field:

$$H = \mathrm{rot}\, A, \quad E = -\frac{1}{c}\frac{\partial A}{\partial t}.$$

Let us now consider the equation

$$\left\{\left(\frac{i\hbar}{c}\frac{\partial}{\partial t}\right)^2 - \left(-i\hbar\frac{\partial}{\partial x} + \frac{e}{c}A\right)^2 - m^2c^2 - \frac{\hbar e}{c}(\sigma, H) + i\varrho_1\frac{\hbar e}{c}(\sigma, E)\right\} Z(x,t) = 0. \tag{3}$$

In our case

$$A = a\cos\varphi, \quad H = -\frac{2\pi\nu}{c}[a, n]\sin\varphi\,^{2)}, \quad E = \frac{2\pi\nu}{c}a\sin\varphi.$$

If the notation

$$-\frac{2\pi\nu\hbar e}{c^2}\{(\sigma,[a,n]) + i(\alpha, a)\} = -\frac{2\pi\nu\hbar i e}{c^2}[\alpha n + 1](\alpha, a) = g$$

is introduced, the sign is changed in (3), and account is taken of the fact that $\sum_{k=1}^{3}\frac{\partial}{\partial x_k}a_k\cos\varphi = 0$ (since $(a, n) = 0$), equation (3) for the given field may

[1] The difference consists only in the sign standing before the brackets.
[2] $[a, n]$ is the sign of the vectorial product of a and n.

also be written as follows:

$$\left\{\frac{\hbar^2}{c^2}\frac{\partial^2}{\partial t^2} - \hbar^2 \frac{\partial}{\partial x_k^2} - 2 i \hbar \frac{e}{c} a_k \cos\varphi \frac{\partial}{\partial x_k} + \frac{e^2}{c^2} a_k^2 \cos^2\varphi + m^2 c^2 + g \sin\varphi\right\} Z(x,t) = 0. \tag{4}$$

In this equation summation is to be made according to k in every term containing k.

Let p be the vector of the moment of motion of the free electron and E its energy. Then we have:

$$\frac{E^2}{c^2} = p_1^2 + p_2^2 + p_3^2 + m^2 c^2 = p^2 + m^2 c^2. \tag{5}$$

We assume that the solution of equation (4)[1] may be represented in the form

$$Z(x,t) = e^{-\frac{i}{\hbar}[Et - px] + F(\varphi)}. \tag{6}$$ [1]

where F (ϕ) is a required matrix function which is to have the following property: $FF' = F'F$, F'; F' designates the derivation of function F (ϕ) according to ϕ.

If we differentiate (6) we obtain:

$$\frac{\partial Z}{\partial t} = \left[-\frac{iE}{\hbar} + 2\pi\nu F'\right] Z,$$
$$\frac{\partial^2 Z}{\partial t^2} = \left\{\left[-\frac{iE}{\hbar} + 2\pi\nu F'\right]^2 + 4\pi^2\nu^2 F''\right\} Z,$$
$$\frac{\partial Z}{\partial x_k} = \left[\frac{i p_k}{\hbar} - \frac{2\pi\nu n_k}{c} F'\right] Z,$$
$$\frac{\partial^2 Z}{\partial x_k^2} = \left\{\left[\frac{i p_k}{\hbar} - \frac{2\pi\nu n_k}{c} F'\right]^2 + 4\pi^2\nu^2 n_k^2 F''\right\} Z;$$

[1] e = Napier.

F'' denotes the second derivative of function $F(\phi)$ according to ϕ.

We now have

$$\frac{\hbar^2}{c^2}\frac{\partial^2 Z}{\partial t^2} - \hbar^2 \sum_{k=1}^{3} \frac{\partial^2 Z}{\partial x_k^2} = -\left\{m^2 c^2 + \frac{4\pi\nu\hbar i}{c}\left[\frac{E}{c} - np\right]F'\right\}Z.$$

The terms containing the second derivative F'' will vanish, since $h_1^2 = h_2^2 + h_3^2 = 1$. We also have

$$-2i\hbar\frac{e}{c}\cos\varphi \sum_{k=1}^{3} a_k \frac{\partial Z}{\partial x_k} = \left[2\frac{(a,p)\cdot e}{c}\cos\varphi\right]Z.$$

Equation (4) yields

$$\left\{-\frac{4\pi\nu\hbar i}{c}\left[\frac{E}{c} - np\right]F' + \frac{2(a,p)e}{c}\cos\varphi + \frac{e^2 a^2}{c^2}\cos^2\varphi + g\sin\varphi\right\}Z = 0;$$

hence

$$F' = \frac{1}{4\pi\nu\hbar i\left[\frac{E}{c} - nP\right]}\left[gc\sin\varphi + 2(a,p)e\cos\varphi + \frac{a^2 e^2}{2c}(1+\cos 2\varphi)\right],$$

and

$$F(\varphi) = \frac{1}{4\pi\nu\hbar i\left[\frac{E}{c} - np\right]}\left[-gc\cos\varphi + (a,p)e\sin\varphi + \frac{a^2 e^2(2\varphi + \sin 2\varphi)}{4c\left[\frac{E}{c} - np\right]}\right]$$

+ a constant matrix.

Now $Z\ (x,\ t)$, which satisfies equation (4), has the form

$$Z(x,t) = e^{-\frac{i}{\hbar}\left[Et - px\right] \frac{-gc}{4\pi\nu\hbar i\left[\frac{E}{c} - np\right]}\cos\varphi + \frac{(a,p)e}{2\pi\nu\hbar i\left[\frac{E}{c} - np\right]}\sin\varphi + \frac{a^2 e^2 (2\varphi + \sin 2\varphi}{16\pi\nu\hbar i c\left[\frac{E}{c} - np\right]}} \cdot Z^0. \tag{7}$$

Z^0 is an arbitrary constant matrix which in general may also depend on the constants p_1, p_2, and p_3. (Z^0 is introduced because an arbitrary constant maxtrix is contained in ~~the~~ expression for $F\ (\phi)$.)

Matrix g_1 in the exponents has the following property:

$$g_1 = \frac{-gc}{4\pi\nu\hbar i\left[\frac{E}{c} - np\right]} = \frac{e(1+\alpha n)(\alpha, a)}{2c\left[\frac{E}{c} - np\right]},$$

$$g_1^2 = \frac{e^2}{4c^2\left[\frac{E}{c} - np\right]^2}(1+\alpha n)(\alpha, a)(1+\alpha n)(\alpha, a)$$

$$= \frac{e^2}{4c^2\left[\frac{E}{c} - np\right]^2}(1+\alpha n)(1-\alpha n)(\alpha, a)^2 = 0,$$

since $\quad (1+\alpha n)(1-\alpha n) = 1 - (\alpha, n)^2 = 1 - 1 = 0.$

If the expression obtained for $Z\ (x;\ t)$ is expanded into a series according to the powers of g_1 and the fact is taken into account that all these powers save the first equal zero, $Z\ (x;\ t)$ may be written in the following form:

$$Z(x,t) = (1 + g_1\cos\varphi)\, e^{-\frac{i}{\hbar}[Et - px] - \frac{i}{\hbar}S},$$

where

$$S = \frac{ape}{2\pi\nu\left[\frac{E}{c} - np\right]}\sin\varphi + \frac{a^2 e^2}{16\pi\nu c\left[\frac{E}{c} - np\right]}(2\varphi + \sin 2\varphi). \tag{8}$$

ist.

If operator (2) is applied to the function $Z\ (x;\ t)$, we obtain ϕ_p, which will satisfy Dirac equation (1):

$$\psi_p = \left\{\frac{i\hbar}{c}\frac{\partial}{\partial t} - \left(\alpha, -i\hbar\frac{\partial}{\partial x} + \frac{e}{c}A\right) - \varrho_3 m c\right\} Z(x,t)$$

$$= \left\{\frac{(1-\alpha n)\, g}{2\left[\frac{E}{c} - n p\right]}\sin\varphi + \frac{(1-\alpha n)\, a p e}{c\left[\frac{E}{c} - n p\right]}\cos\varphi + \frac{(1-\alpha n)\, a^2 e^2}{2c^2\left[\frac{E}{c} - n p\right]}\cos^2\varphi \right.$$

$$\left. - \frac{e}{c}(\alpha, a)\cos\varphi + \frac{E}{c} - \alpha p - \varrho_3 m c\right\} Z(x,t).$$

The last expression is simplified if Z is adopted in the form (8) and the following relations are taken into account:

$$(1-\alpha n)\, g = 0, \quad (1-\alpha n)\, g_1 = 0.$$

$$-\frac{e}{c}(\alpha, a)\cos\varphi\, g_1 \cos\varphi = -\frac{e^2(\alpha,a)(1-\alpha n)(\alpha,a)}{2c^2\left[\frac{E}{c} - n p\right]}\cos^2\varphi = -\frac{(1-\alpha n)\, a^2 e^2}{2c^2\left[\frac{E}{c} - n p\right]}\cos^3\varphi.$$

After conversion we obtain

$$\psi_p = \left\{\frac{e}{c\left[\frac{E}{c} - n p\right]}\left[(1-\alpha n)\, a p - (\alpha, a)\left(\frac{E}{c} - n p\right)\right.\right.$$

$$\left.\left. + \tfrac{1}{2}\left(\frac{E}{c} - \alpha p - \varrho_3 m c\right)(1+\alpha n)(\alpha, a)\right]\cos\varphi + \frac{E}{c} - \alpha p - \varrho_3 m c\right\} e^{-\frac{i}{\hbar}[Et - px] - \frac{i}{\hbar}S} . Z.$$

After certain modifications we can transfer the factor $\left(\frac{E}{c} - \alpha p - \varrho_3 m c\right)$ from right to left by taking the following equations into account.

$$(\alpha, p)(\alpha, \alpha) = a, p) + i(\sigma, [p, a]),$$

$$(\alpha, p)(\alpha, a) = -(\alpha, a)(\alpha, n),$$

$$(\alpha, p)(\alpha, a) = -(\alpha, a)(\alpha, p) + 2(a, p),$$

$$(\alpha, a)(\alpha, n)(\alpha, p) = i\varrho_1(p_1[a, n]) + (\alpha, [p, [a, n]])$$

$$(\alpha, p)(\alpha, a)(\alpha, n) = (\alpha, a)(\alpha, n)(\alpha, p) - 2(\alpha, [p, [a, n]])$$

$$\frac{1}{2}\left(\frac{E}{c} - \alpha p - \varrho_3 m c\right)(\alpha, a)(\alpha n) = \tfrac{1}{2}(\alpha, a)(\alpha, n)\left(\frac{E}{c} - \alpha p - \varrho_3 m c\right) - (\alpha, [p, [a, n]]),$$

$$(\alpha, a)(n, p) - (\alpha n)(a, p) = (\alpha, [p, [a, n]]).$$

The following expression will then be obtained for the brackets situated with cos ϕ as a factor:

$$\frac{e}{c\left[\frac{E}{c}-np\right]}\Big[(\alpha,a)(n,p)-(\alpha,n)(a,p)+ap-(\alpha,a)\frac{E}{c}$$
$$+\frac{(\alpha,a)}{2}\left(\frac{E}{c}-\alpha p-\varrho_3 mc\right)-ap-\frac{(\alpha,a)(\alpha n)}{2}\left(\frac{E}{c}-\alpha p-\varrho_3 mc\right)$$
$$-(\alpha,[p,[a,n]])\Big]=\frac{-e(\alpha,a)(1+\alpha n)}{2c\left[\frac{E}{c}-np\right]}\left(\frac{E}{c}-\alpha p-\varrho_3 mc\right).$$

We set

$$\frac{e(\alpha,a)(1+\alpha n)}{2c\left[\frac{E}{c}-np\right]}=g_1^{+}. \tag{9}$$

This matrix, g_1^{+}, has the following property: $(g_1^{+})^2 = 0$. Hence ϕ_p $(x;\, t)$ may finally be written in the following two formulas:

$$\psi_p=(1-g_1^{*}\cos\varphi)\,e^{-\frac{i}{\hbar}S}\left(\frac{E}{c}-\alpha p-\varrho_3 mc\right)e^{-\frac{i}{\hbar}[Et-px]}\,\psi^0$$
$$=(1-g_1^{+}\cos\varphi)\,e^{-\frac{i}{\hbar}S}\,\psi^{0\prime}. \tag{10}$$

where

$$\left.\begin{aligned}\psi_p&=e^{\gamma}\,\psi^{0\prime}\\ \gamma&=-g_1^{+}\cos\varphi-\frac{i}{\hbar}S\end{aligned}\right\} \tag{11}$$

and

$$\psi^{0\prime}=\left(\frac{E}{c}-\alpha p-\varrho_3 mc\right)\psi^0\,e^{-\frac{i}{\hbar}[Et-px]}. \tag{12}$$

ist.

If everywhere in the preceding operations we set $a_k = 0$, and we are justified in so doing, since we had to perform no division by a_k, we obtain $\gamma = 0$; $\psi^{0\prime}$ is thus the solution of equation (1) for the case in which no field is present. Since ψ is usually assumed to consist of a single column, we may, as we have already done in the last formulas, also replace Z^0 by ψ^0, which then may consist of a single column (say the first). As is to be desired, the integral (7) found depends on an arbitrary real vector p, which has the meaning of the vector of the moment of motion of the electron still not influenced by the field. We consider in space p a point p_0 with a small environment Δp_0, where p_0 may be an internal point of Δp_0; ψ_p may then be normalized in the usual fashion so that the following relation is satisfied:

$$\lim_{|\Delta p_0 \to 0|} \frac{1}{|\Delta p_0|} \int \left[\int_{\Delta p_0} \psi_p^+ \, dp \int_{\Delta p_0} \psi_p \, d_p \right] dx = 1, \tag{13}$$

$|\Delta p_0|$ denotes the volume of Δp_0, x traverses the entire space, and ψ_p^+ is the transported matrix conjugate to ψ_p;

$$dx = dx_1 \, dx_2 \, dx_3, \quad dp = dp_1 \, dp_2 \, dp_3;$$

each integral sign must here be conceived of as being replaced by a triple sign.

Formula (11) shows that ψ_p is a product of two periodic functions with different periods $1/\nu$ and $1/V$, where V is the frequency related to E by the relation $E + \dfrac{a^2 e^2}{4c\left[\dfrac{E}{c} - np\right]} = hV$. If the ratio ν/V is rational, this product as well is a certain periodic function; in the continuum of E values the cases of the latter kind form an everywhere dense enumerable set.

2. Superposition of Electromagnetic Waves with Different Frequencies and Initial Phases.

We now proceed to investigate the case in which the scalar potential A_0 equals zero, while the vector potential is of the form

$$A = a \sum_{j=1}^{\infty} b^j \cos 2\pi \nu^j \left[t - \frac{nx}{c} + \alpha^j \right] = \sum_{j=1}^{\infty} a^j \cos \varphi^j \tag{14}$$

In this equation a and n are unit vectors perpendicular to each other.

$$\varphi^j = 2\pi\nu^j\left[t - \frac{nx}{c} + \alpha^j\right], \quad a^j = ab^j,$$

ν^j, α^j, b^j denote arbitrary real numbers, the series $\sum_{j=1}^{\infty} |b^j|$ converging.

We shall see later that the enumerable set of frequencies ν^j also cannot be completely arbitrary; it rather must be subject to certain general conditions, this being just as necessary for convergence of the subsequently arising infinite series as is now the absolute convergence of $\sum_{j=1}^{\infty} a^j$.

For the potentials considered (14) equation (4) may be written in the form

$$\left\{\left(\frac{i\hbar}{c}\frac{\partial}{\partial t}\right)^2 - \left(-i\hbar\frac{\partial}{\partial x_k} + \frac{e}{c}\sum_{j=1}^{\infty} a_k^j \cos\varphi^j\right)^2 - m^2c^2 - \frac{\hbar e}{c}(\sigma, \vec{H}) + i\varrho_1\frac{\hbar e}{c}(\sigma, \vec{E})\right\} Z = 0 \tag{15}$$

(It is here necessary to sum up from one to three according to κ.) $\vec{H}$ is the magnetic and $\vec{E}$ the electromagnetic fields associated with potentials (14).

We introduce the following notation:

$$\begin{aligned} g_1^j &= \frac{e}{2c\left[\frac{E}{c} - np\right]}(1 + \alpha n)(\alpha, a^j), \\ \gamma^j &= g_1^j \cos\varphi^j + \frac{(a^j, p)\, e}{2\pi\nu^j\hbar i\left[\frac{E}{c} - np\right]}\sin\varphi^j + \frac{a^2e^2(2\varphi^j + \sin 2\varphi^j)}{16\pi\nu^j\hbar ic\left[\frac{E}{c} - np\right]} \\ &\quad - \frac{2\pi\nu^j\hbar ie}{c^3}(1 + \alpha n)(\alpha, a^j) = g^j. \end{aligned} \tag{16}$$

It is to be noted that the matrices g_1^j equal each other except for a constant factor b^j; hence they commutate among themselves and according to the foregoing have the property $g_1^j g_1^{j'} = 0$, which is valid both for $j \neq j'$ and for $j = j'$. For this reason we may count to the right with the matrices g_1^j as with ordinary numbers.

If the sign of the left-and side of equation (15) is changed and it is noted that

$$\frac{\hbar e}{c}(\sigma, \vec{H}) - i\varrho_1 \frac{\hbar e}{c}(\sigma, \vec{E}) = \sum_{j=1}^{\infty} g^j \sin \varphi^j$$

is valid, the equation in question may be written in the form

$$\begin{aligned} \Big\{ \frac{\hbar^2}{c^2}\frac{\partial^2}{\partial t^2} - \hbar^2 \frac{\partial^2}{\partial x_k^2} - 2i\hbar \frac{e}{c} a_k^j \cos \varphi^j \frac{\partial}{\partial x_k} \\ + \frac{e^2}{c^2} a_k^j a_k^{j'} \cos \varphi^j \varphi^{j'} + m^2 c^2 + g^j \sin \varphi^j \Big\} Z - 0 \end{aligned} \tag{17}$$

(in this case summation is to be made according to k from one to three and according to j and j' from one to ∞; j and j' change independently of each other). We shall henceforth everywhere dispense with the summation sign and merely write out the common term, in each instance with index, always indicating how the indices are changed.

We first assume that

$$Z = e^{-\frac{i}{\hbar}[Et - px] + \sum_{j=1}^{\infty} \gamma^j} \cdot Z^0 \tag{17*}$$ [1]

---

[1] We here assume absolute and uniform convergence of the series $\Sigma\gamma^j$, this imposing certain restrictions on the aggregate of the frequencies; however, the required convergence will of itself be fulfilled on the basis of conditions to which this aggregate of frequencies subsequently must at any rate be subjected.

we introduce it into equation (17), and then perform the operations indicated there; when this is done the exponenial function may be differentiated in the usual manner, since the exponent commutates with its derivative. Only the following terms accordingly remain on the left-hand side.

$$\frac{e^2}{c^2}\left[\sum_{j,j''}(a^j, a^{j''}) \cos\varphi^j \cos\varphi^{j''}\right] Z = \sum_{j,j''} q^{j,j''} \cdot Z = q. \tag{17**}$$

j and j'' independently traverse the value from one to ∞; the only requirement is that $j'' \neq j$. (The other terms vanish in accordance with the relation $n_1^2 + n_2^2 + n_3^2 = 1$. .)

So that the terms written out in (17**) will also vanish, we further set

$$Z = e^{-\frac{i}{\hbar}[Et - px] + \gamma^j + \gamma^{jj''}} \cdot Z^0. \tag{18}$$

(In the exponent it is necessary to perform summation according to j, j'' from one to ∞; $j \neq j''$). For the pair jj'', in which $\nu^j \neq \nu^{j''}$ is to be found, we assume that

$$\gamma^{jj''} = \mu^{jj''} \cos\varphi^j \sin\varphi^{j''} + l^{jj''} \sin\varphi^j \cos\varphi^{j''}, \tag{17***}$$

where $\mu^{jj''}$ and $l^{jj''}$ are to be independent of x and t. Substitution of (18) and (17) yields the factors of Z in addition to $q^{jj''}$:

$$\left.\begin{aligned} &2\left\{-\mu^{jj''}\frac{2\pi\nu^{j''}\hbar i}{c}\left[\frac{E}{c}-np\right] \quad \frac{2\pi\nu^j\hbar i}{c}\left[\frac{E}{c}-np\right]\right\} \cos\varphi^j \cos\varphi^{j''}, \\ &2\left\{\mu^{jj''}\frac{2\pi\nu^j\hbar i}{c}\left[\frac{E}{c}-np\right] + l^{jj''}\frac{2\pi\nu^{j''}\hbar i}{c}\left[\frac{E}{c}-np\right]\right\} \sin\varphi^j \sin\varphi^{j''}. \end{aligned}\right\} \tag{18*}$$

The remaining terms vanish on the basis of the relation $+ n_3^2 = 1.$
If we select the constants $\mu^{jj''}$ and $l^{jj''}$ as follows

$$\left.\begin{aligned} l^{jj''} &= -\frac{\nu^j}{\nu^{j''}}\,\mu^{jj''}, \quad \frac{4\pi\hbar i}{c}\left[\frac{E}{c} - np\right](\mu^{jj''}\nu^{j''} + e^{jj''}\nu^{j}) = \frac{e^2}{c^2}(a^j, a^{j''}), \\ \mu^{jj''} &= \frac{e^2}{4\pi\hbar i c\left[\frac{E}{c} - np\right]}\cdot\frac{(a^j, a^{j''})\,\nu^{j''}}{[(\nu^{j''})^2 - (\nu^j)^2]} \end{aligned}\right\} \qquad (19)$$

and subject the aggregate of frequencies ν^j to such restrictions that the series

$$\sum_{j,\,j''=1}^{\infty}\left|\frac{b^j b^{j''}\nu^{j''}}{(\nu^{j''})^2 - (\nu^j)^2}\right|,$$

converges, after substitution of (18) and (17) we obtain simply zero on the left-hand side. The second term in (18*) vanishes as a result of the special choice of $l^{jj''}$, and the first term likewise vanishes as a result of the choice of $\mu^{jj''}$, which here equals the number $q^{jj''} = \frac{e^2}{c^2}(a^j, a^{j''})$ with the opposite sign. The series considered converge absolutely and uniformly.

However, if $\nu^{j''} = \nu^j$, the associated term in (17**) is

$$\begin{aligned} &\frac{e^2}{c^2}(a^j a^{j''})\cos 2\pi\nu^j\left[t - \frac{nx}{c} + \alpha^j\right]\cos 2\pi\nu^j\left[t - \frac{nx}{c} + \alpha^{j''}\right] \\ &\qquad = \frac{e^2}{2c^2}(a^j a^{j''})\cos(\varphi^j + \varphi^{j''}) + \frac{e^2}{2c^2}(a^j, a^{j''})\cos(\alpha^j - \alpha^{j''}). \end{aligned}$$

as may readily be seen, we may in this case assume

$$\gamma^{jj''} = \frac{(a^j, a^{j''})\,e^2}{16\,\nu^j\hbar i c\left[\frac{E}{c} - np\right]}\left[(\varphi^j + \varphi^{j''})\cos(\alpha^j - \alpha^{j''}) + \sin(\varphi^j + \varphi^{j''})\right] \qquad (20)$$

If we set $j'' = j$ in the last formula, we obtain in the expression for γ^j the last term

$$\frac{(a^j)^2 e^2}{16\pi\nu^j\hbar i c\left[\frac{E}{c} - np\right]}[2\varphi^j + \sin 2\varphi^j];$$

hence the latter may be regarded as γ^{jj}.

ψ_p, which satisfies equation (1) under the given potential conditions of (14) for the external field, will now be of the following form:

$$\psi_p = \left\{\frac{i\hbar}{c}\frac{\partial}{\partial t} - \left(\alpha, -i\hbar\frac{\partial}{\partial x} + \frac{e}{c}\sum_{j=1}^{\infty} a^j \cos\varphi^j\right) - \varrho_3 mc\right\} e^{-\frac{i}{\hbar}[Et - px] + \sum_{j=1}^{\infty} B^j \cos\varphi^j + \sum_{j=1}^{\infty} C^j \sin\varphi^j + \sum_{jj'} \gamma^{jj'}} \cdot \psi^0. \tag{21}$$

In this equation

$$\gamma^{jj'} = \mu^{jj'}\cos\varphi^j \sin\varphi^{j'} + l^{jj'}\sin\varphi^j \cos\varphi^{j''},$$

if $j' \neq j$ and $\varphi^j - \varphi^{j'} \neq \text{const.}$, and

$$\gamma^{jj'} = R^{jj'}(\varphi^j + \varphi^{j'}) + K^{jj'}\sin(\varphi^j + \varphi^{j'}),$$

if $j = j'$ or $\phi^j - \phi^{j'} = $ const., i.e., $\nu^{j'} = \nu^j$; j and j' vary independently of each other from one to $+\infty$; in addition,

$$B^j = g_1^j,\quad C^j = \frac{(a^j, p)\, e}{2\pi\nu^j\hbar i\left[\frac{E}{c} - np\right]},\quad K^{jj'} = \frac{e^2 (a^j, a^{j'})}{16\pi\nu^j\hbar i c\left[\frac{E}{c} - np\right]}. \tag{21*}$$

$$R^{jj'} = K^{jj'}\cos(\alpha^j - \alpha^{j'}).$$

If we make use of the properties of the matrices initially referred to, we may also write expression (21) in the form

$$\psi_p = \left\{\frac{i\hbar}{c}\frac{\partial}{\partial t} - \left(\alpha, -i\hbar\frac{\partial}{\partial x} + \frac{e}{c}\sum_{j=0}^{\infty} a^j \cos\varphi^j\right) - \varrho_3 mc\right\}\left(1 + \sum_{j=1}^{\infty} g_1^j \cos\varphi^j\right) e^{-\frac{i}{\hbar}[Et - px] + \sum_{j=1}^{\infty} C^j \sin\varphi + \sum_{j,j'} \gamma^{jj'}} \cdot \psi^0 \tag{22}$$

By superposition of such solutions with different values of p we obtain for the solutions of the Dirac equation, under the given special conditions, a still more general explicit expression

$$\psi = \int c(p)\, \psi_p \, \mathrm{d}p. \tag{23}$$

Again, the integral in this case is to be considered as being a triple integral. $c\ (p)$ denotes any desired matrix, one subject only to the condition that integral (23) exist if the range of integration is infinite.

We hope to be able to continue the foregoing operations and give certain applications in a future paper.

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